

Geometrical consequences of foam equilibrium

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Equilibrium conditions impose nontrivial geometrical constraints on the configurations that a two-dimensional foam can attain. In the first place, the three centers of the films that converge to a vertex have to be on a line, i.e., all vertices are *aligned*. Moreover, an equilibrated foam must admit a *reciprocal figure*. This means that it must be possible to find a set of points P_i on the plane, one per bubble, such that the segments $\overline{P_i P_j}$ are normal to the corresponding foam films. It is furthermore shown that these constraints are equivalent to the requirement that the foam be a sectional multiplicative Voronoï partition (SMVP). A SMVP is a cut with a two-dimensional plane of a three-dimensional multiplicative Voronoï partition. Thus, given an arbitrary equilibrated foam, we can always find pointlike sources (one per bubble) in three dimensions that reproduce this foam as a generalized Voronoï partition. These sources are the only degrees of freedom that we need in order to describe the foam fully. [S1063-651X(97)14405-1]

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I. INTRODUCTION

Cellular structures [1–4] appear in a wide range of natural phenomena, and have puzzled and fascinated scientists for decades [5]. They can be generally described as packings of space-filling cells of roughly polygonal shape, separated by thin interfaces to which a surface energy is associated. They arise as a consequence of competition between domains under the constraint of space filling, and there is often some mechanism, such as the migration of a conserved quantity across the interface, which makes them evolve in time, i.e., coarsen. Foams such as those obtained by shaking soapy water are the examples of cellular structures closest to our everyday experience. Two-dimensional foams may be obtained by confining a soap froth between two closely spaced glass plates. In spite of their apparent simplicity, foams display much of the phenomenology appearing in coarsening cellular structures. Foams have been the subject of interest since their relevance in the problem of grain growth was pointed out by Smith [6]. Despite the attention they have received, the understanding of their dynamical properties proved to be a tricky problem. Even some of the most basic issues, such as their asymptotic scaling properties, has been a matter of debate until recently (see references above). Numerically exact simulation procedures [7], as well as analytical [8] and numerical [9] approximations, have been useful in understanding the dynamics of ideal foams, but the system sizes accessible to available simulation procedures are strongly limited (for a complete account, see Refs. [3,4]).

A satisfactory theoretical framework for the description of foams has not yet been achieved. The main result of this work consists in establishing a *rigorous connection* between foams and Voronoï partitions (VP's). This connection provides a set of fundamental degrees of freedom for the foam (the source's locations in space), and therefore constitutes a step toward the above-mentioned theoretical understanding.

On the other hand, an efficient method for the numerical simulation of ideal foams would also be highly desirable, and we propose that such a method could be obtained by exploiting the equivalence between foams and VP's reported in this work. More precisely, we establish the following *correspondence* between equilibrated two-dimensional foams (EF's) and a generalization of VP's, the sectional multiplicative Voronoï partition (SMVP): *Given an arbitrary EF, it is always possible to find sources in space such that a SMVP with respect to them exactly gives this foam.*

In Sec. II the VP and its generalizations are reviewed. The simplest generalizations of the VP concept are the sectional Voronoï partition and the multiplicative Voronoï partition. These correspond to adding a constant to the (square) distance, and to multiplying it by a constant, respectively. At the end of this section a combination of both is introduced, the sectional multiplicative Voronoï partition. We will use this partition in order to describe foams.

The demonstration of the above-mentioned correspondence between foams and SMVP's is divided in two parts (Secs. III and IV) for clarity. In Sec. III, the *recognition problem* for SMVP's is solved. The recognition problem consists in giving the sufficient geometrical conditions that a circular partition has to satisfy in order to be a SMVP. We will see that if a circular partition has all of its vertices *aligned* and admits an *oriented reciprocal figure*, then it is a SMVP. In other words, it is always possible to find sources in three-dimensional space that give this circular partition as a SMVP. In this section we also describe the procedure to *find* the sources when these conditions are met. Some material that is needed in this section is described in the Appendixes.

Section IV deals with the equilibrium conditions for foams and its geometrical consequences. We start by writing the force and pressure equilibrium conditions in compact form in Sec. IV A. In Secs. IV B and IV C, it is shown that an equilibrated foam has all of its vertices *aligned* and admits an *oriented reciprocal figure*, respectively. Therefore all equilibrated foams satisfy the conditions required in Sec. III,

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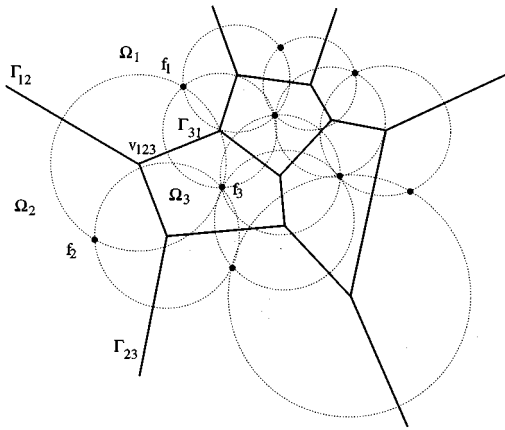


FIG. 1. Shown is a Voronoi partition of two-dimensional space with respect to pointlike sources $\{f_i\}$ (black dots). Each source f_i has an associated cell Ω_i , which is the subset of space that is closer to f_i than to any other source. Interfaces Γ_{ij} (thick lines) are equidistant from the sources whose cells they delimit. Three interfaces Γ_{ij} , Γ_{jk} , and Γ_{ki} meet at a vertex v_{ijk} , which is therefore the center of a circle (dashed) through the three corresponding sources.

and can be described as SMVP's. Finally in Sec. V the implications of this result are discussed, and some perspectives for future work are advanced.

II. SPACE PARTITIONS

A. Voronoi partition

Given a set of N pointlike sources $\{f_i\}$ in n space, the *Voronoi partition* or *tessellation* of space with respect to $\{f_i\}$ is a classification of space into *cells* Ω_i defined such that $x \in \Omega_i$ if x is closer to f_i than to any other source [10],

$$\Omega_i = \{x \in R^n / d(x, i) < d(x, j), \quad \forall j \neq i\}, \quad (1)$$

where $d(x, i)$ denotes the distance between x and source f_i . This construction is also known under the names Wigner-Seitz cells, Dirichlet tessellations, and Thiessen polygons. In two dimensions, the dual lattice of a Voronoi partition is called the Delaunay triangulation. Cell Ω_i can be seen as the ‘‘region of influence’’ of f_i , where the sources compete for some spatially distributed resource. It is common to use these constructions, and their generalizations, as *approximate* models for cellular structures occurring in nature [10–13].

Two neighboring cells Ω_i and Ω_j are delimited by an *interface* Γ_{ij} of points x equidistant from f_i and f_j ,

$$\Gamma_{ij} = \{x \in R^n / d(x, i) = d(x, j) < d(x, k), \quad \forall k \neq i, j\}. \quad (2)$$

These interfaces are $(n-1)$ -dimensional hyperplanes normal to $f_i f_j$. In two dimensions Ω_i are convex polygons and the interfaces Γ_{ij} are straight lines (Fig. 1). Three interfaces Γ_{ij} , Γ_{jk} , and Γ_{ki} meet at a *vertex* v_{ijk} , which is equidistant from f_i , f_j , and f_k . This means that v_{ijk} is the center of a circle through f_i , f_j , and f_k . Vertices of higher multiplicity are possible for particular locations of the sources. For example, a fourfold vertex would exist if four sources lay on the same circle. We will ignore the existence of these par-

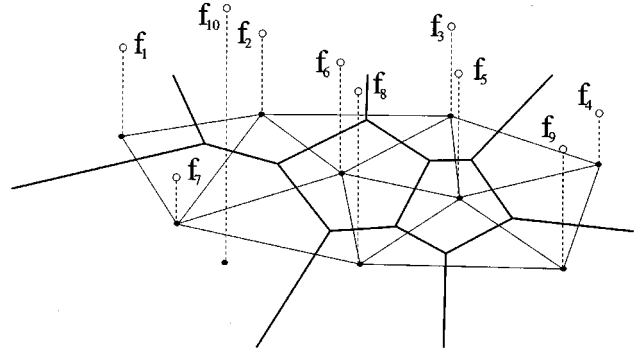


FIG. 2. A sectional Voronoi partition (SVP) of the plane is defined by intersecting a three-dimensional Voronoi partition with that plane. Sources P_i of this SVP (black dots) are the projections of the three-dimensional sources f_i (white dots) onto the plane. Interfaces Γ_{ij} between cells are rectilinear, normal to $P_i P_j$, and meet at triple vertices v_{ijk} , but are *not* equidistant from the sources. Sources with a too large value of z_i (for example P_{10} in this figure) do not have an associated cell.

ticular configurations, that is, we will assume *generic* source locations. Under this assumption, all vertices are of multiplicity three.

The concept defining Voronoi partitions is *equidistance*. A simple way to generalize them is to change the way in which distances to the sources are measured. One lets each source measure the distance to points x according to its own ‘‘rule.’’ The interface Γ_{ij} is then the set of points x for which two neighbors ‘‘claim’’ the same distance. Two simple ways to do this are by adding an arbitrary constant to the square of the distance (sectional Voronoi partition [10,14,15]) and by multiplying the distance by a constant (multiplicative Voronoi partition [10,16,15,17]).

B. Sectional Voronoi partitions

A *sectional Voronoi partition* (SVP) is defined [10,14,15] as a k -dimensional cut of a higher-dimensional Voronoi partition. The sources $\{P_i\}$ of the sectional partition are defined as the projections of the original sources $\{f_i\}$ onto this lower-dimensional k space. Cell Ω_i associated to source P_i is the intersection of cell $\tilde{\Omega}_i$ (associated to f_i) with this k -dimensional hyperplane. For example, take sources $\{f_i\}$ in three dimensions (Fig. 2), and construct a VP with three-dimensional cells Ω_i and plane interfaces $\tilde{\Gamma}_{ij}$. Now project the sources $f_i = (x_i, y_i, z_i)$ onto the $z=0$ plane (which we call the Π_z plane) to obtain the projected sources $P_i = (x_i, y_i)$, which have associated ‘‘heights’’ z_i . Assign to each source P_i the intersection Ω_i of $\tilde{\Omega}_i$ with Π_z . This defines the a sectional partition of Π_z with respect to sources $\{P_i\}$ with heights z_i . Cells Ω_i are thus defined as

$$\Omega_i = \{x \in \Pi_z / d^2(x, i) + z_i^2 < d^2(x, j) + z_j^2, \quad \forall j \neq i\}, \quad (3)$$

where $d(x, i)$ is the distance, on the Π_z plane, between P_i and point x .

In the same way, interfaces Γ_{ij} are defined as

$$\Gamma_{ij} = \{x \in \Pi_z / d^2(x, i) + z_i^2 = d^2(x, j) + z_j^2 < d^2(x, k) + z_k^2, \quad \forall k \neq i, j\}. \quad (4)$$

Properties of these partitions in two dimensions are the following: (a) Interfaces are straight lines and (generically) meet at triple vertices. (In dimension d , vertices have multiplicity $d+1$.) (b) Interface Γ_{ij} is normal to $\overline{P_i P_j}$, but not, in general, equidistant from P_i and P_j . (c) The partition is unchanged if all z_i are changed according to: $z_i^2 \rightarrow z_i^2 + c^2$ for c an arbitrary real. An important difference from the Voronoi partition is that in this case there may be sources to which no cell is associated. This happens if $\tilde{\Omega}_i$ is not cut by Π_z . It is easy to see that SVP's are *equivalent* to Laguerre partitions [14].

Appendix A discusses the *recognition problem* for this kind of partitions, that is, giving the sufficient conditions for an arbitrary rectilinear partition to be a SVP. We will use similar concepts in Sec. III in order to solve the recognition problem for sectional multiplicative Voronoi partitions.

As an example of the application of sectional partitions, consider the case of sources f_i , whose location in space changes in time, giving rise to a time-dependent partition of Π_z . If, for example, source f_1 moves away from Π_z , the two-dimensional cell Ω_1 will shrink and finally disappear when $\tilde{\Omega}_1$ no longer cuts Π_z . Therefore the number of cells can vary without changing the number of sources. Thus sectional partitions can be used as dynamical models for crystal growth [13], and other processes in which some cells disappear or are created.

C. Multiplicative Voronoi partition

The multiplicative Voronoi partition (MVP) [16,15,17] is defined by assigning to each source f_i a positive *intensity* a_i , and defining the *multiplicative distance* $d_m(x, i) = d(x, i)/a_i$. The cell Ω_i associated to source f_i is then the set of points x closer to f_i (in terms of this multiplicative distance) than to any other source

$$\Omega_i = \left\{ x \in R^n / \frac{d(x, i)}{a_i} < \frac{d(x, j)}{a_j}, \quad \forall j \neq i \right\}. \quad (5)$$

Interfaces Γ_{ij} are hyperspherical surfaces (circle arcs in two dimensions) satisfying

$$\Gamma_{ij} = \left\{ x \in R^n / \frac{d(x, i)}{a_i} = \frac{d(x, j)}{a_j} < \frac{d(x, k)}{a_k}, \quad \forall k \neq i, j \right\}. \quad (6)$$

In two dimensions, the circular interface of a MVP with two sources is one of the Apollonius circles (see for example [18]) of those two points. In arbitrary dimensions let d_{ij} be the distance between two sources f_i and f_j , and $A_{ij} = a_j/a_i$. Without loss of generality, we take $A_{ij} < 1$. This means that f_j has the smallest intensity and therefore Ω_j will be the interior of a hypersphere, while Ω_i will be its exterior. The following properties are satisfied in any dimension: (a) f_i is contained in Ω_i ; (b) Γ_{ij} has radius $R_{ij} = A_{ij} d_{ij} / (1 - A_{ij}^2)$, and its center C_{ij} is on the straight line going

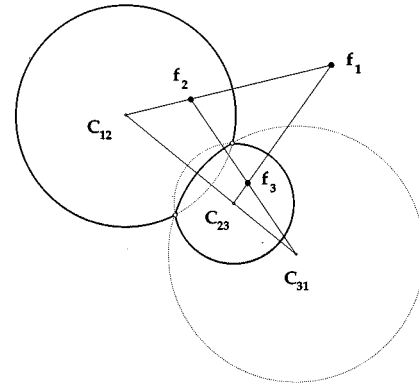


FIG. 3. A multiplicative Voronoi partition with respect to three sources in two dimensions. The source with the largest intensity a_i (source f_1 in this figure) has an unbounded cell. The other two sources are within their associated cells. Angles formed by the interfaces at the vertices depend on the positions of the sources and the values of the intensities. For certain values of $\{a_i\}$ the interfaces may not intersect. In this case the partition would be composed of two disjoint circles each containing one of the sources, with smaller intensities. The MVP with respect to three sources in three-dimensional space is a cluster of two spherical bubbles (Fig. 4, dashed lines).

through f_i and f_j ; and (c) C_{ij} is located at a distance $d_{ij}/(1 - A_{ij}^2)$ from f_i , that is, it never lays between f_i and f_j .

Three sources give rise to a tessellation like the one shown in Fig. 3. The exterior of the two ‘‘bubbles’’ is the infinite cell associated to the source with the largest intensity, f_1 in this example. For some choices of $\{a_i\}$ the interfaces will not intersect. In this case the tessellation is simply a pair of circles, each containing one of the sources with smaller intensities, while the exterior of these two circles is the cell of the source with larger intensity. MVP's can be interpreted again as ‘‘areas of influence’’ of sources f_i , but now each source has a different strength.

Consider a MVP of two-dimensional space with respect to a set of sources $\{f_i\}$. If three interfaces Γ_{ij} , Γ_{jk} , and Γ_{ki} meet at a vertex (again, vertices of higher multiplicity are only possible for particular configurations, which we ignore here), then centers C_{ij} , C_{jk} , and C_{ki} lay on a line. The reason why this is so is simple. If two interfaces Γ_{ij} and Γ_{jk} have a common point v_{ijk} , then their *continuations* must meet again at another point v_{ijk}^* , the *conjugate vertex*. But then the third interface Γ_{ik} must also go through this point since $d(v^*, i)/a_i = d(v^*, j)/a_j$ and $d(v^*, j)/a_j = d(v^*, k)/a_k$ implies $d(v^*, i)/a_i = d(v^*, k)/a_k$. Therefore two points exist (v and v^*) at which all three circles intersect. Then the centers of these circles must be on a line (Fig. 3).

Let us define a *circular partition* (CP) of two-dimensional space to be a classification of space in cells delimited by arbitrary circle arcs that meet at triple points called vertices. No three centers of these arbitrarily defined interfaces will in general be on a line. We will say that a vertex of a CP is *aligned* if the centers of the three interfaces defining it are on a line. As we saw, all vertices of a MVP are aligned in two dimensions (2D). Therefore for each triple vertex of a MVP

in 2D, the three sources and the three centers form the configuration [19] $(6_2, 4_3)$ of projective geometry. Since C_{ij} cannot be between P_i and P_j , the three centers can only be on one of the two external segments of the configuration.

In a three-dimensional MVP, four cells meet at each vertex v_{ijkl} , giving rise to six spherical interfaces (ij) , (ik) , (il) , (jk) , (jl) , and (kl) . The centers of these interfaces are aligned in triplets so that the six centers also form the configuration $(6_2, 4_3)$, this time in three-dimensional space. This has the implication that the six centers are necessarily on the same plane.

The MVP was first introduced by Boots [16] to describe areas of influence in geography. Ash and Bolker [15] also studied the recognition problem, that is, under which conditions a CP is a MVP. The visual similarity between this kind of partitions and two-dimensional foams is striking, and suggests the idea of finding a connection between them. Obviously the centers of the films of a foam must be aligned for each triple vertex, if the foam is to be described as a MVP, since MVP's are aligned. One finds [17] that this alignment condition is indeed satisfied by all vertices of arbitrary equilibrated foams. Despite this (which is a necessary but not sufficient condition for a foam to be a MVP), two-dimensional MVP's cannot describe all possible equilibrated foams in two dimensions. This work shows the reason of this limitation: In order to describe arbitrary equilibrated two-dimensional foams we must introduce the sectional variant of a multiplicative partition. In other words, instead of confining the sources to the plane, we let them exist in a three-dimensional space, and obtain the foam as a *two-dimensional cut of a three-dimensional MVP*.

D. Sectional multiplicative Voronoï partition

We will restrict the description to the case of a plane cut of a three-dimensional MVP. The generalization to a k -dimensional cut of an n -dimensional MVP is straightforward. A plane cut, with a plane Π_x , of a three-dimensional MVP defines a SMVP of Π_z (see Figs. 4 and 5). The sources P_i of this SMVP are the normal projections onto Π_z of the original sources f_i , and may be seen to have as attributes both an intensity a_i and a height z_i . Cells Ω_i associated to sources P_i are defined as

$$\Omega_i = \left\{ x \in \Pi_z \mid \frac{d(x,i)^2 + z_i^2}{a_i^2} < \frac{d(x,j)^2 + z_j^2}{a_j^2}, \quad \forall j \neq i \right\}. \quad (7)$$

In the same way, interfaces Γ_{ij} are circle arcs satisfying

$$\Gamma_{ij} = \left\{ x \in \Pi_z \mid \frac{d^2(x,i) + z_i^2}{a_i^2} = \frac{d^2(x,j) + z_j^2}{a_j^2} < \frac{d^2(x,k) + z_k^2}{a_k^2}, \right. \\ \left. \forall k \neq i, j \right\}. \quad (8)$$

An example of a SMVP with three sources is shown in Figs. 4 and 5. We see that there are two vertices on Π_z at which the three interfaces meet. In a general case (for example, in a partition with respect to many sources like in Fig. 6), if vertex v_{ijk} exists, then the continuations of interfaces Γ_{ij} , Γ_{jk} , and Γ_{ki} also meet at a *conjugate* vertex v_{ijk}^* . This im-

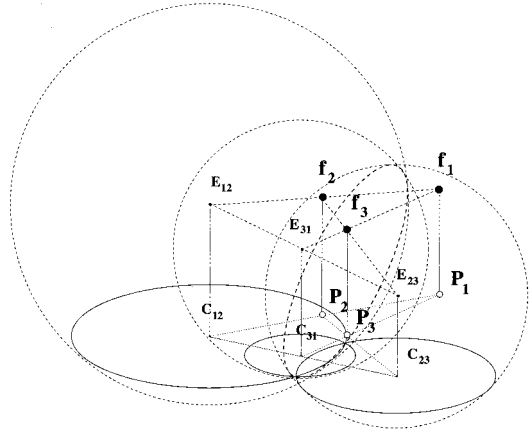


FIG. 4. A sectional multiplicative Voronoï partition (SMVP) is defined as a plane section (black circles) of a three-dimensional multiplicative Voronoï partition (dashed lines). Shown is an example for three sources. Sources P_i of the SMVP (on the Π_z plane, white dots) are the projections of sources f_i of the MVP in three dimensions (black dots). The centers E_{ij} of the 3D MVP are aligned, which implies the alignment of the centers C_{ij} of the SMVP. Segments $\overline{f_i f_j}$ are normal to spheres $\tilde{\Omega}_{ij}$; therefore $P_i P_j$ are normal to circles Γ_{ij} on the plane.

plies the alignment of centers C_{ij} , C_{jk} , and C_{ki} , which could also be deduced from the alignment of E_{ij} , E_{jk} , and E_{ki} in three dimensions (Fig. 4).

We notice that the SMVP is equivalent to a multiplicative Laguerre partition, since a SVP is equivalent to a Laguerre partition [14]. As is usual in sectional partitions, in the SMVP there may be sources P_i with no associated cell, those for which the corresponding three-dimensional ‘‘bubble’’ $\tilde{\Omega}_i$ does not cut Π_z .

It was seen in Sec. II C that the spherical interface $\tilde{\Gamma}_{ij}$ in three dimensions is cut normally by the segment $\overline{f_i f_j}$. As a projective consequence of this, the straight line containing

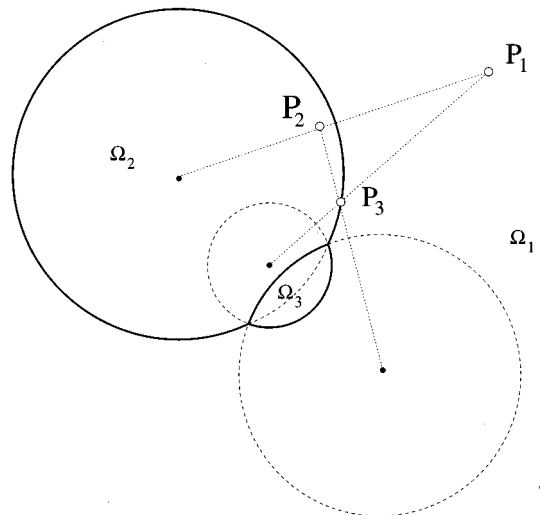


FIG. 5. An upper view of the SMVP with respect to three sources defined in Fig. 4. Note that sources P_i are not all contained in their cell Ω_i , as happens in the case of the MVP (Fig. 4).

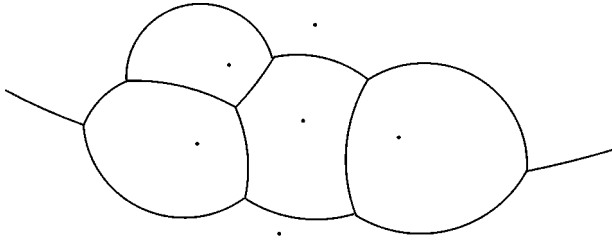


FIG. 6. A SMVP of the plane with six sources. The projections P_i of the sources are indicated as small dots in the figure. For each one of the vertices in this partition, the three interfaces can be continued and they will meet each other again at a *conjugate* vertex.

P_i and P_j on Π_z is normal to the circular interface Γ_{ij} . In other words, P_i , P_j , and C_{ij} are on the same line. Furthermore, and since E_{ij} never lays between f_i and f_j in 3D, we notice that C_{ij} is always outside the segment $\overline{P_i P_j}$.

III. WHEN A CIRCULAR PARTITION IS A SMVP

In this section the sufficient conditions for a CP to be a SMVP are given. We first introduce the notion of oriented reciprocal figure for CP's, and then proceed to demonstrate that an aligned CP admitting such a reciprocal figure is a SMVP.

A. Reciprocal figure of a circular partition

We will now generalize the concept of reciprocal figure as appropriate for circular partitions. We say that a graph \mathcal{R} made of points P_i connected by edges (ij) forms a reciprocal figure for a CP \mathcal{P} if (i) points P_i are in correspondence with the cells Ω_i of \mathcal{P} ; (ii) edges (ij) are in correspondence with the interfaces Γ_{ij} of \mathcal{P} ; or (iii) for each Γ_{ij} in \mathcal{P} , points P_i , P_j , and C_{ij} lay on the same line. Consider two regions Ω_i and Ω_j separated by a circular interface Γ_{ij} . A reciprocal figure \mathcal{R} will be said to respect *orientation* if for each Γ_{ij} in \mathcal{P} ; (i) C_{ij} is not between P_i and P_j ; and (ii) starting from C_{ij} and traveling along $C_{ij}P_iP_j$ to infinity, points P_i and P_j are found in the same order as regions Ω_i and Ω_j .

We saw already that all SMVP's are aligned (all vertices satisfy the alignment condition introduced in Sec. II C). On the other hand, it is clear that the sources P_i of a SMVP form a reciprocal figure that satisfies orientation. Therefore, if \mathcal{P} is a SMVP then (i) it is aligned, and (ii) it admits an oriented reciprocal figure

Figure 7 shows four possible partitions that share the same centers C_{12} , C_{23} , and C_{31} . All four admit reciprocal figures. But only in cases (a) and (c) do we see a reciprocal figure that satisfies orientation can be constructed. Therefore neither (b) nor (c) can be a SMVP. The reason for this is that the orientation condition cannot be satisfied if any of the vertices is not *convex*. A vertex is *convex* if all the internal angles formed by the interfaces are less than π . Convexity of all vertices is clearly a necessary condition for a circular partition to be a SMVP. We are now ready to give the *sufficient* conditions for a circular partition to be a SMVP. We will show in Sec. III B that if a circular partition \mathcal{P} is aligned and admits a reciprocal figure that satisfies orientation, then it is a SMVP.

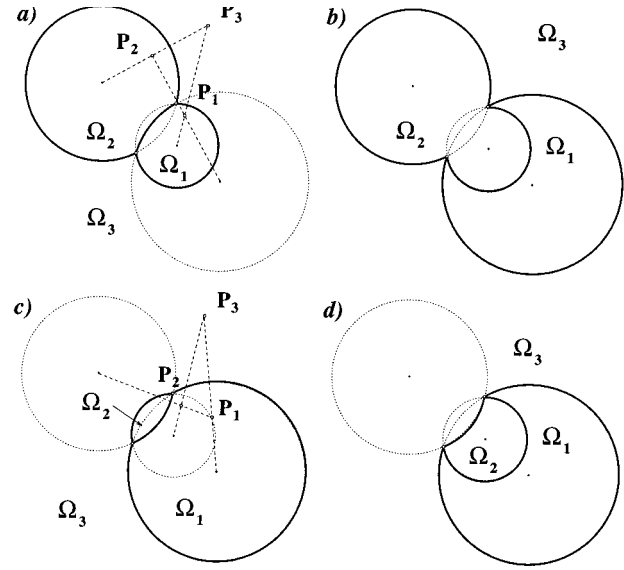


FIG. 7. All these four circular vertices admit a reciprocal figure. Two of the vertices, (a) and (c), are convex, and therefore a reciprocal figure satisfying orientation is possible. The other two are not convex, so that no reciprocal figure can satisfy orientation on them.

B. Finding the sources of a SMVP

Given a circular partition \mathcal{P} of a region of two-dimensional space, all of whose (triple) vertices are aligned, we show here that, if \mathcal{P} admits a reciprocal figure that satisfies orientation, then \mathcal{P} is a SMVP with respect to sources f_i located somewhere above points P_i . Our demonstration is constructive, that is, we explicitly show how the sources and intensities are determined. For this purpose we will use the inversion transformation (Appendix B) to *straighten* each vertex in turn, that is, transform a circular vertex into a rectilinear one. This rectilinear vertex is one of a SVP. Sources are located in this *straight* representation (Appendix A), and then transformed back to the original system. The intensities a_i of the corresponding SMVP are obtained in this back-transformation, since the SMVP is invariant under inversion.

Let v be an aligned vertex on which three interfaces $\{\Gamma_{12}, \Gamma_{23}, \Gamma_{32}\}$ meet, and $\{P_1, P_2, P_3\}$ the three points of the reciprocal figure associated to the three bubbles sharing the vertex, as in Fig. 8. The conjugate vertex v^* is obtained as the intersection point of the *continuations* of the interfaces, which happens at a unique point because of alignment. Let furthermore λ_i be the straight line through v^* and P_i . We start by showing the following:

Theorem 1. A monoparametric family of triplets of circles $\{\omega_{12}, \omega_{23}, \omega_{31}\}$ exists, which has the following properties: (a) ω_{12} , ω_{23} , and ω_{31} intercept each other on v^* . (b) $\omega_{ij} \perp \Gamma_{ij}$. (c) The intersection point q_i between ω_{ij} and ω_{ik} lays on λ_i .

The nontrivial content of the theorem is the fact that the three intersection points q_i of these normal circles ω_{ij} lay on the lines λ_i . For example take an arbitrary point q_1 on λ_1 and draw a circle ω_{12} through q_1 and v^* , and normal to Γ_{12} . Let q_2 be the intersection of ω_{12} with λ_2 . Now draw a circle ω_{23} through q_2 and v^* , and normal to Γ_{23} . Let q_3 be its intersection with λ_3 . Then the circle ω_{31} through v^* , q_1 , and q_3 is normal to Γ_{13} .

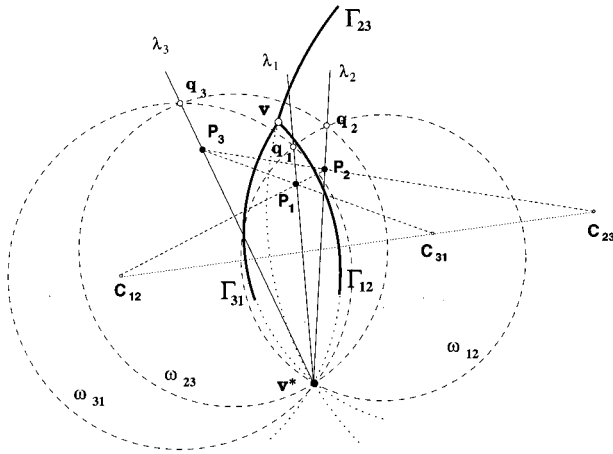


FIG. 8. A vertex v with its reciprocal figure $P_1P_2P_3$, showing circles ω_{ij} . These circles are normal to the interfaces Γ_{ij} , and intersect each other at points q_i on λ_i .

Proof. The demonstration is done by first performing an inversion with center v^* and arbitrary radius, whereupon the interfaces Γ_{ij} are transformed into straight lines Γ'_{ij} meeting at v' (Fig. 9). This transformed system will be referred to as the “straight” representation of that vertex. Primed names are used in the straight representation, with the exception of v^* , whose original location is kept in Fig. 9 (v^* itself is mapped to ∞ by the inversion). Lines originally not going through v^* are transformed into circles through v^* , as is the case of $\{S_{12}, S_{23}, S_{31}\}$, which form the reciprocal figure in the original system. Circles ω_{ij} are transformed into straight lines ω'_{ij} going through q'_i and q'_j , respectively, on λ_i and λ_j , which are invariant.

The inversion transformation preserves angles, therefore circle S'_{ij} is normal to the now straight interface Γ'_{ij} . This means that its center L_{ij} must lay on Γ'_{ij} . Consider the figure formed by v', L_{12}, L_{23} , and L_{31} , and the six lines joining them. Maxwell [20] showed that a figure made of four points joined by six lines *always* has a reciprocal figure (to see this,

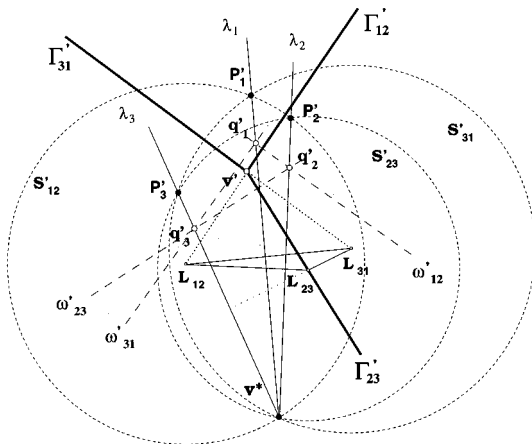


FIG. 9. Same as Fig. 8, after an inversion around v^* . All circles originally through v^* are now straight lines. This is the case of interfaces Γ_{ij} , and therefore we call this the “straight” representation of the vertex.

just consider the centers of the four circles going through three of these points). We will now identify it on Fig. 9 as the figure formed by v^*, q'_1, q'_2 , and q'_3 . First we notice that circles S'_{ij} and S'_{jk} intercept each other at two points v^* and P'_j , both on λ_j . Therefore lines $L_{ij}L_{jk}$ are normal to λ_j , and we have identified the first three lines of the reciprocal figure. The other three lines $\omega_{ij} = q'_i q'_j$ of the reciprocal figure have to be normal to the segments vL_{ij} , so they are normal to the interfaces Γ'_{ij} . Since the inversion preserves angles, this means that $\omega_{ij} \perp \Gamma_{ij}$ in the original system (Fig. 8).

We see that the existence of triplets of circles $\{\omega_{12}, \omega_{23}, \omega_{31}\}$ with the above-mentioned properties is a consequence of the existence, in the straight representation, of a figure $v^*q'_1q'_2q'_3$, which is the reciprocal of $vL_{12}L_{23}L_{31}$. In other words, theorem 1 means that in the straight representation, triangle $q'_1q'_2q'_3$ forms a reciprocal figure for the rectilinear vertex. Clearly this reciprocal figure $q'_1q'_2q'_3$ satisfies orientation if the original figure $P_1P_2P_3$ satisfies orientation in the circular system. Therefore as discussed in Appendix A, it is always possible to find three sources $f'_i, i = 1, 2$, and 3 , located at heights z'_i above points q'_i , such that this rectilinear vertex is a SVP with respect to them. Once these sources f'_i are known in the straight representation, a second inversion around the same point v^* takes us back to the original system, and provides the original locations f_i of the sources.

If v_{123} is the first vertex to be considered in the system, we have one degree of freedom in the determination of the circles ω_{ij} , or equivalently of points q_i . Once in the straight representation, there is one more degree of freedom: the determination of *one* of the heights z'_i . We will now show that, if this z'_i is chosen appropriately, the three back-transformed sources f_i are located above their respective points P_i .

Consider a plane Π_i that is normal to Π_z and contains λ_i , as shown in Fig. 10. Draw a circle β_i through q_i and v^* , and normal to Π_z . Let u_i be the intersection of this circle with the normal α_i to Π_z , through P_i . Now under an inversion with center v^* , point u'_i is above q'_i , since circle β_i is transformed in a straight line β'_i , normal to Π_z . Straight line α_i normal to Π_z , through P_i is now a circle α'_i through v^* and u'_i , and normal to Π_z . Its intersection with Π_z determines the image P'_i . Now we now that source f'_i is located somewhere on β'_i . The sources can be displaced vertically (simultaneously) according to what we see in Appendix A, but not independently. Fixing the height z'_i of one of them determines the other two uniquely. What we want to demonstrate is that if one of the sources f'_i coincides with its point u'_i , then the other two also do. In order to do this, take, for example, the sphere $\tilde{\mathcal{X}}_{12}$ through v^*, f'_1 , and f'_2 , and normal to Π_z . In the straight representation, interface $\tilde{\Gamma}'_{12}$ of the Voronoi partition, with respect to sources f'_1 and f'_2 is a plane, equidistant from f'_1 and f'_2 , and normal to $f'_1f'_2$. This means that $\tilde{\Gamma}'_{12}$ is normal to $\tilde{\mathcal{X}}_{12}$. As a consequence, the intersection \mathcal{X}_{12} (a circle) of $\tilde{\mathcal{X}}_{12}$ with Π_z is normal to the intersection Γ'_{12} (a straight line) of $\tilde{\Gamma}'_{12}$ with Π_z . Now assume that $f'_1 = u'_1$. This means that $P'_1 \in \tilde{\mathcal{X}}_{12}$, since α'_1 is in this case contained in $\tilde{\mathcal{X}}_{12}$. But then the circle

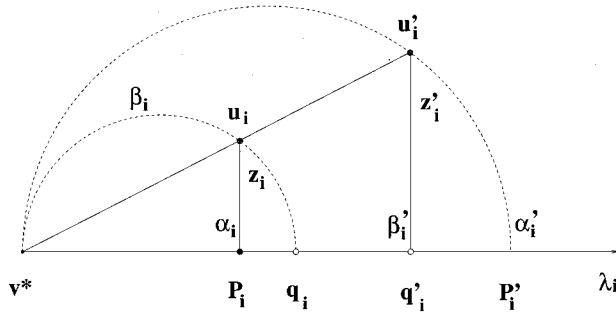


FIG. 10. Shown is a plane normal to Π_z and containing λ_i . This figure shows how the height z_i of source $f_i = u_i$ in the original system is determined, given the positions of P_i and q_i . Primed variables correspond to positions after the inversion, that is, in the ‘‘straight’’ system. Sources $f'_i = u'_i$ are above q'_i , and therefore the back-transformed sources f_i are above P_i .

\mathcal{X}_{12} is the circle S'_{12} (Fig. 9) that goes through P'_1 , P'_2 , and v^* , and is normal to Γ'_{12} . This implies that P'_2 is also in $\tilde{\mathcal{X}}_{12}$, which in turn implies $f'_2 = u'_2$. The same reasoning can be repeated for the pair of sources 1 and 3.

We have thus shown that all sources f_i are, in the back-transformed, original system, located above the points P_i of the reciprocal figure. In other words, we have shown that the given reciprocal figure is the projection of the sources f_i on the Π_z plane.

Now we have to determine the heights z_i and intensities a_i . The heights z_i of the sources can be found by back-transforming the sources f'_i from the straight system. However in practice this is not necessary. By looking at Fig. 10 we notice that triangle $v^* f_i q_i$ is a rectangle. Therefore,

$$z_i^2 = |v^* P_i| \times |P_i q_i|, \quad (9)$$

which suffices to locate the sources f_i with the sole knowledge of points P_i and q_i in the original system. Knowing now the spatial locations of the sources f_i in three dimensions, the intensities a_i follow immediately, since the vertex v^* is equidistant [21] from the three sources. This means

$$a_i = A |v^* f_i| = A (|v^* P_i| \times |v^* q_i|)^{1/2}. \quad (10)$$

Here A is an arbitrary positive constant. Alternatively we can obtain the same result by using the transformation properties [Eq. (B2)] of the a_i 's, and the fact that the intensities are all equal in the straight system.

We have used the inversion transformation to demonstrate that the given vertex is a SMVP with respect to sources f_i located above P_i , and showed how the sources f_i can be located without the need to perform an inversion for each vertex explicitly. All we need is to know the location of points q_i , which we are free to choose for the first vertex under consideration. This first choice determines all subsequent q_i points, and therefore all sources. It is clear from Fig. 10 that $|v^* q_i| \geq |v^* P_i|$ has to be satisfied. Therefore one has to choose the triplet of circles w_{ij} such that this condition is verified for the three points q_i . If one of the q_i 's coincides with P_i , this means that the corresponding height z_i is zero, i.e., the source f_i lays on the Π_z plane. A point q_i closer to v^* than the corresponding P_i is not acceptable.

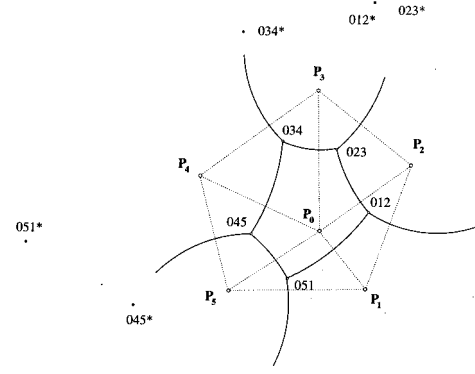


FIG. 11. The points P_i of the reciprocal figure (empty circles) are the projections, on the xy plane, of the sources f_i in three-dimensional space. The lines $P_i P_j$ must thus be normal to interfaces Γ_{ij} . Black dots indicate the location of the conjugate vertices v_{ijk}^* for each vertex v_{ijk} in the figure.

The general procedure to determine the sources can now be described. Assume we are given a circular tessellation which satisfies the required conditions of alignment and convexity, and that we are able to find, or are given, a reciprocal figure P_i for it. We start from one of the cells of the partition, as in Fig. 11. Take an arbitrary vertex to start with, for example, vertex v_{012} , and determine tentative locations for point q_0 on λ_0 . This fixes q_1 and q_2 , as discussed in Sec. III B. After fixing q_0 , the rest of the construction is uniquely determined. The locations q_0 , q_1 , and q_2 determine z_0 , z_1 , and z_2 , and the corresponding intensities a_0 , a_1 , and a_2 , through Eqs. (9) and (10). Now proceed to the neighboring vertex v_{023} . Two of the sources, f_0 and f_2 are already known, only the height z_3 of f_3 has to be found. First one has to find the *new* locations of q_0 and q_2 when defined from vertex v_{023}^* . In order to do this, lines λ_i^{023} from vertex v_{023}^* are drawn, and q_0 and q_2 are located using the fact that the heights z_0 and z_2 are already known. Equation (9) implies

$$|P_0 q_0^{(023)}| = |P_0 q_0^{(012)}| \frac{|v_{(012)}^* P_0|}{|v_{(023)}^* P_0|}, \quad (11a)$$

$$|P_2 q_2^{(023)}| = |P_2 q_2^{(012)}| \frac{|v_{(012)}^* P_2|}{|v_{(023)}^* P_2|}. \quad (11b)$$

Once the new q_0 and q_2 are known, the next step is to draw circles ω_{03} and ω_{23} through them and normal to the respective interfaces. Their intersection gives the location of q_3 (this intersection always occurs on λ_3^{023} , as theorem 1 shows). This determines z_3 and a_3 using Eqs. (9) and (10). This procedure of triangulation is repeated until all sources are located. Eventually it may happen that a point q_i is found to be closer to the conjugate vertex v^* than the corresponding P_i . This is not acceptable, and means that the *tentative* starting position of q_0 has to be changed. It must be shifted away from v_{012}^* , and the whole procedure repeated. It is easy to see that taking a large enough value of q_0 always solves this problem.

One could ask whether this construction can be closed self-consistently. For example, after determining f_5 from f_0

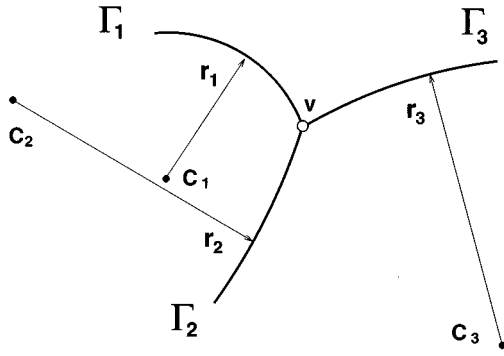


FIG. 12. A generic vertex v , at which three interfaces Γ_1 , Γ_2 , and Γ_3 meet. We regard the pressure drops to be positive if the pressure decreases when the interface is crossed in the counterclockwise sense of rotation around v . Therefore, in the case shown in this figure, r_2 and r_3 are positive, while r_1 is negative.

and f_4 in our example of Fig. 11, one can go on with the procedure as if f_1 were unknown. Would the position of f_1 determined by f_0 and f_5 be the same one as found initially? Alternatively, if we used f_0 and f_1 to determine f_5 instead of going around the bubble in the opposite sense, would its position be the same as found after going around the bubble? The answer is yes, because of unicity. As discussed in Appendix A, the height of one of the sources of a vertex determines the other two uniquely. This means that f_1, f_2, \dots, f_5 are all uniquely determined by f_0 .

IV. EQUILIBRATED FOAMS

In this section we show that a two-dimensional foam in equilibrium satisfies all conditions required for it to be a SMVP. In order to do this, let us first write down the equilibrium conditions for an arbitrary vertex of the foam in compact form. We will consider the case of foams with arbitrary surface tensions, and also allow forces to act on the foam's vertices.

A. Equilibrium equations

Let \vec{v} be the location of a vertex v at which three interfaces Γ_1 , Γ_2 , and Γ_3 meet, as shown in Fig. 12. Each interface Γ_i is a circle arc with radius r_i and center \vec{C}_i . It produces, on v , due to its surface tension τ_i , a force \vec{T}_i of modulus $2\tau_i$ in the direction of the tangent to the film at \vec{v} . These forces \vec{T}_i can therefore be written as

$$\vec{T}_i = -2\frac{\tau_i}{r_i}\vec{K}_i^* = -\xi_i\vec{K}_i^*, \quad (12)$$

where $\vec{K}_i = \vec{C}_i - \vec{v}$ and $\xi_i = 2\tau_i/r_i$. The asterisk here stands for a counterclockwise rotation in $\pi/2$, so that $\vec{e}_x^* = \vec{e}_y$. Let us more generally assume that an external force \vec{F} acts on v . Equilibrium of all forces acting on the vertex implies

$$\vec{F}^* + \sum_{i=1}^3 \xi_i \vec{K}_i = \vec{0}, \quad i=1,2,3. \quad (13)$$

There is still one more condition, which is related to pressure equilibrium around the vertex. The pressure drop across an interface can be written as

$$\Delta p_i = 2\frac{\tau_i}{r_i} = \xi_i. \quad (14)$$

We will adopt the convention for Δp to be positive if the pressure decreases when crossing the interface in the counterclockwise sense of rotation around v . This is of course related to a convention for the signs of the r_i . In Fig. 12, r_1 and Δp_1 are negative according to this convention. Notice that pressure jumps and radii have different signs when considered from the two opposite vertices joined by a film. The fact that the total accumulated pressure drop around a vertex has to be zero then implies

$$\sum_{i=1}^3 \xi_i = 0. \quad (15)$$

This is the pressure equilibrium condition for the vertex. Equations (13) and (15) are satisfied if the vertex is equilibrated, and together are equivalent to

$$\vec{F}^* + \sum_{i=1}^3 \xi_i (\vec{C}_i - \vec{x}) = \vec{0}, \quad (16)$$

where \vec{x} is an arbitrary point. This is what we will call the equilibrium condition for the vertex, and it encloses both force and pressure equilibrium.

B. Equilibrium implies alignment

The alignment of the centers of a two-dimensional foam was already shown a long time ago by Plateau [22] for equal surface tensions, and in the case of small self-standing clusters of two and three bubbles. For the case of a cluster of two bubbles and zero load, it is a trivial consequence of symmetry [23]. The demonstration for the case of clusters of three bubbles is referred to by Boys as being ‘‘rather long and difficult’’ [22].

It is not difficult to see that the alignment of the centers is in no way a property of clusters, and also not restricted to vertices with equal surface tensions and zero loads, but a general consequence of equilibrium. We will find that, under very general conditions, such as arbitrary surface tensions and external loads applied on the vertices, if a vertex is equilibrated then the centers of the three arcs converging to it lay on a line.

Consider Eq. (16), and assume for a moment that the external load is zero. Then C_1 , C_2 , and C_3 lay on the same line, as can be seen by taking $\vec{x} = \vec{C}_1$, for example. Thus all vertices in equilibrium are aligned if no external force is applied on it. This alignment property is even true under nonzero load conditions, if the force is perpendicular to the line of centers. A load satisfying such condition will be called a *normal load*. The alignment condition has the geometrical consequence that the interfaces Γ_i , when continued, meet each other again at a unique point v^* which we called the *conjugate vertex* [24] of v . This also means that the vertex v could be physically realized as a self-standing cluster of

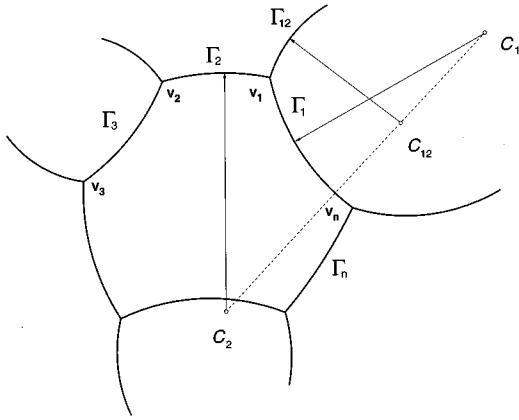


FIG. 13. A bubble surrounded by n neighbors $\alpha=1, \dots, n$, showing the naming convention for the interfaces. Note that film curvatures are of opposite signs when considered from each of the two vertices at their ends. Thus, for example, ξ_2 appears with a $+$ sign in the equilibrium condition of vertex v_2 , but with a $-$ sign in that of v_1 [see Eq. (19)].

two bubbles (by continuing its interfaces), and is a kind of a “separability” condition for the static equilibrium conditions, in the sense that each vertex of a foam might as well be that of an isolated cluster of two bubbles. Thus a vertex in equilibrium under a normal load is aligned, or, equivalently, a vertex in equilibrium under a normal load has a conjugate vertex.

Next we would like to consider the existence of a reciprocal figure, since this is also a condition that has to be satisfied in order for a foam to be a SMVP. This condition must be separately considered. The reader may easily build examples of bubbles with n neighbors, all of whose n vertices are aligned, but yet do not admit a reciprocal figure. The reason is that the attempted reciprocal figure will not in general “close” around that bubble, the same case as described in Appendix A.

C. Equilibrium implies reciprocal figure

We will now show that if all vertices of a foam are equilibrated, then an oriented reciprocal figure exists for it. We start by considering a bubble, and show that the reciprocal figure can be found for it. The figure for the whole foam can then be formed by patching together those of neighboring bubbles. Consider a bubble with n neighbors $\alpha=1, 2, \dots, n$, as shown in Fig. 13. At each vertex v_α of this bubble, three films Γ_α , $\Gamma_{\alpha+1}$, and $\Gamma_\alpha^{\alpha+1}$ meet. Interface Γ_α separates the central bubble from its neighbor α , while interface $\Gamma_\alpha^{\alpha+1}$ is the limit between neighbors α and $(\alpha+1)$. We will assume vertices v_α to be in equilibrium under arbitrary *normal* loads \vec{F}_α . We can generally write these loads as

$$\vec{F}_\alpha^* = -R_{\alpha+1}\vec{C}_{\alpha+1} + S_\alpha\vec{C}_\alpha + T_\alpha\vec{C}_\alpha^{\alpha+1}. \quad (17)$$

That is, we have decomposed the external load of vertex v_α in the *dependent* basis formed by the three centers of the films meeting at the vertex. Because these centers are

aligned, and the load is normal to the line of centers, the following condition is always satisfied by the coefficients:

$$-R_{\alpha+1} + S_\alpha + T_\alpha = 0. \quad (18)$$

The equilibrium condition (16) now reads

$$\begin{aligned} & -(\xi_{\alpha+1} + R_{\alpha+1})(\vec{C}_{\alpha+1} - \vec{x}_\alpha) + (\xi_\alpha + S_\alpha)(\vec{C}_\alpha - \vec{x}_\alpha) \\ & + (\xi_\alpha^{\alpha+1} + T_\alpha)(\vec{C}_\alpha^{\alpha+1} - \vec{x}_\alpha) = \vec{0}, \alpha = 1, \dots, n, \end{aligned} \quad (19)$$

and holds for *arbitrary* x_α 's. The sign of ξ_α is determined by the sign convention at vertex v_α , therefore $\xi_{\alpha+1}$ must appear with a minus sign in the equilibrium equation for vertex α . Since we use a dependent basis [Eq. (17)] the coefficients $\{R_{\alpha+1}, S_\alpha, T_\alpha\}$ (the “representation” of the load) are not uniquely determined for a given load. There is instead a monoparametric family of coefficients $\{R_{\alpha+1}, S_\alpha, T_\alpha\}$, all satisfying both Eqs. (17) and (18), for each load F_α . We will use these degrees of freedom to choose a representation $\{\tilde{R}_{\alpha+1}, \tilde{S}_\alpha, \tilde{T}_\alpha\}$ that satisfies

$$\tilde{S}_\alpha = \tilde{R}_\alpha, \quad \alpha = 1, 2, \dots, n. \quad (20)$$

Notice that this condition relates the coefficients of two consecutive loads. We now show that such a representation always exists. We start by making the degree of freedom in the representation of the loads explicit. For arbitrary v_α , we add the null vector (19) multiplied by v_α to the load (17), and obtain

$$\vec{F}_\alpha^* = -\tilde{R}_{\alpha+1}\vec{C}_{\alpha+1} + \tilde{S}_\alpha\vec{C}_\alpha + \tilde{T}_\alpha\vec{C}_\alpha^{\alpha+1}, \quad (21)$$

where

$$\tilde{R}_{\alpha+1} = R_{\alpha+1} + v_\alpha(R_{\alpha+1} + \xi_{\alpha+1}), \quad (22a)$$

$$\tilde{S}_\alpha = S_\alpha + v_\alpha(R_\alpha + \xi_\alpha), \quad (22b)$$

$$\tilde{T}_\alpha = T_\alpha + v_\alpha(T_\alpha + \xi_\alpha^{\alpha+1}). \quad (22c)$$

Condition (20) then implies

$$v_{\alpha+1} = \frac{R_\alpha - S_{\alpha+1}}{S_{\alpha+1} + \xi_{\alpha+1}} + v_\alpha \frac{R_\alpha + \xi_{\alpha+1}}{S_{\alpha+1} + \xi_{\alpha+1}}, \quad (23)$$

which has always a solution in the generic case.

Using this representation of the loads, we can rewrite the equilibrium condition (19) as

$$\begin{aligned} & -Q_{\alpha+1}(\vec{C}_{\alpha+1} - \vec{x}_\alpha) + Q_\alpha(\vec{C}_\alpha - \vec{x}_\alpha) + E_\alpha(\vec{C}_\alpha^{\alpha+1} - \vec{x}_\alpha) = \vec{0}, \\ & \alpha = 1, \dots, n, \end{aligned} \quad (24)$$

where

$$Q_\alpha = \xi_\alpha + \tilde{R}_\alpha, \quad (25a)$$

$$E_\alpha = \xi_\alpha^{\alpha+1} + \tilde{T}_\alpha. \quad (25b)$$

figure for the whole foam. Each selection of a starting point P_0 and a “scale” $|P_0P_1|$ determines the other points uniquely, therefore there are three degrees of freedom in the determination of the reciprocal figure. Once P_0 and P_1 are chosen, all other points are found as intersections of two lines passing through the centers C_{ij} and one already existing point P_i .

D. Orientation condition

Now we have to show that it is always possible to find, among all possible reciprocal figures, at least one that satisfies orientation as defined in Sec. II D. In the first place, if all the surface tensions are positive, then all films will be under traction, and therefore the vertices will be convex. If there are nonconvex vertices in the foam (which would happen if some of the films are compressed instead of stretched) then we know that it is not possible to satisfy orientation (Fig. 7). Positive surface tensions is thus a necessary condition [25] for the foam to be a SMVP, although their modulus can be arbitrary for each film.

The orientation condition could fail in the first place because a center C_{ij} lays in between two points P_i and P_j . It is always possible to avoid this by choosing P_1 close enough to P_0 . In this way all following points P_i are confined within a (arbitrarily chosen) small region of space where no center is located. This ensures that no center C_{ij} lays between P_i and P_j . Now regarding the second part of the orientation condition (Sec. II D), consider a vertex v_{123} , which is convex and aligned. Two orientations of the triangle $P_1P_2P_3$ are possible, as shown in Figs. 7(a) and 7(c). Note that both constitute reciprocal figures for both vertices, but only one of them satisfies orientation in each case. In our construction of the reciprocal figure for the whole foam, we can decide the orientation of the initial triangle, choosing the one that respects orientation. The question is now if the correct orientation of this starting triangle ensures that of all subsequent ones, whose locations are determined by P_0 and P_1 . To demonstrate that this is indeed the case, we note that the triangle of Fig. 7(a), if considered as a reciprocal figure for the vertex of Fig. 7(c), has all three segments P_iP_j wrongly oriented. The point we want to make is that, if the vertex is convex, there are only two possibilities: either all pairs P_iP_j satisfy orientation, or all are wrongly oriented. Then if one of the pairs of a triangle forming part of a reciprocal figure is correctly oriented, the other two must necessarily also be. This demonstrates that if the starting pair P_0P_1 is chosen with the correct orientation, then all subsequent triangles must be correctly oriented, since they share at least a pair of sources with one preexisting triangle. Therefore in order to ensure correct orientation of the whole figure, it is enough to correctly choose the orientation of the first pair.

V. DISCUSSION

A dissection of space into cells separated by circular interfaces that meet at triple vertices is called a circular partition (CP). A two-dimensional foam therefore defines a circular partition of two-dimensional space. The equilibrium conditions for the foam impose geometrical constraints on this CP. We have shown here that the CP defined by an

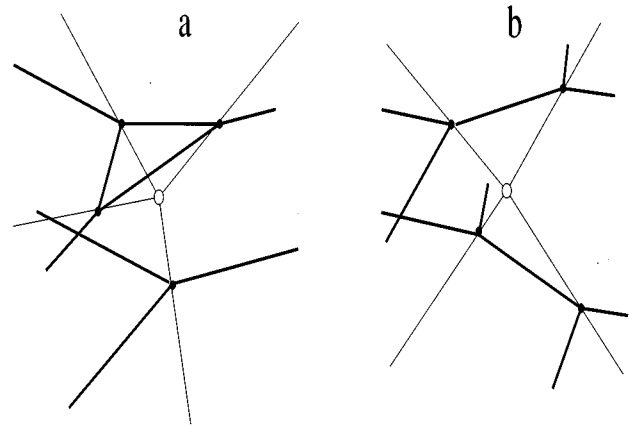


FIG. 15. A three-dimensional multiplicative Voronoï partition gives rise to vertices which simultaneously belong to four neighboring bubbles in a generic situation. Six spherical interfaces converge at these vertices. When one such vertex (white dot) crosses the projection plane Π_z , depending on its orientation this can result either in (a) a T2 process (disappearance of a triangular bubble) or (b) a T1 process (neighbor switching) for the two-dimensional foam.

equilibrated foam is *aligned*, and admits an *oriented reciprocal figure*. This result is valid in general for heterogeneous foams, each of whose films may have an arbitrary (positive) surface tension, and even if loads are applied on the vertices, with the sole requirement of equilibrium. We have seen in Sec. III B that any CP satisfying the conditions of alignment and existence of oriented reciprocal figure is a sectional multiplicative Voronoï partition (SMVP). A SMVP is a plane cut of a multiplicative partition, thus two-dimensional foams are plane cuts of three-dimensional foams, these last being a multiplicative partition with respect to sources in three dimensions. Therefore, given an arbitrary equilibrated two-dimensional foam, it is always possible to find sources $\{f_i\}$ in three-dimensional space, and amplitudes a_i such that the given foam is a SMVP with respect to those sources.

A first implication of this correspondence is the identification of a set of degrees of freedom for the foam: the intensities a_i and locations of the sources f_i in three dimensions. This allows a more natural description of a foam, than the one that is done in terms of films and vertices. When a foam is interpreted as a tessellation of space with respect to some sources, we see that the foam’s films and vertices are secondary constructions, and their evolution is a consequence of that of the sources. The dynamical description is conceptually simpler using the SMVP interpretation. For example, the processes of neighbor switching (T1) and cell disappearance (T2) are described in a unified manner (Fig. 15). Both are due to the fact that a fourfold vertex in 3D crosses the projection plane Π_z . Depending on the spatial orientation of the vertex with respect to Π_z , this is seen as a T1 or T2 process.

An evolving foam can be seen as a particular instance of a dynamical random lattice [26], in which the evolution of a cellular structure is fixed by assigning a given dynamics to the sources of a mathematically defined tessellation. In the case of foams the dynamics is usually fixed by gas diffusion across the membranes. Alternatively, other dynamical evolution rules may also be interesting, but in any case one has to

make the translation to obtain the source's dynamics. The next step is then to find, for a given proposed dynamic evolution for the foam, the corresponding dynamics [27] for the sources and intensities.

Foams are usually studied inside a bounded region or cage, which imposes the only constraint that the films be normal to its boundary. Boundaries of this kind do not affect the fact that the foam is a SMVP. We have shown that the result holds with the sole requirement of vertex equilibrium. It is obvious that SMVP's are always closed on themselves, forming a self-standing cluster, that is, there are no open film ends. This implies that even bounded foams *must* be a region of a larger self-standing cluster of bubbles that is closed on itself, and everywhere equilibrated. The point is not trivial in that it ensures that all films ending at the boundary can be continued, eventually forming (phantom) vertices, and that the resulting foam will have all its vertices in equilibrium. We see then that there is no fundamental difference between bounded and self-standing foams, since all foams are regions of a self-standing cluster. This does not mean that the boundaries have no effect, which would of course be false. If the foam's dynamic inside the cage is, for example, produced by gas diffusion across the films, the evolution of the "phantom" bubbles existing as continuations of the physical foam outside the cage, will *not* follow this dynamic, but a different one, which is determined by the constraint that the films be normal to the cage's boundaries.

In the field of joint-bar structures, an old result due to Maxwell [20,28] states that if a lattice accepts a reciprocal figure then it can support a self-stress, and conversely. More recently Ash and Bolker [15] showed that the existence of a reciprocal figure is sufficient condition for the lattice to be a sectional Voronoï partition. In this case there is the additional requirement that all vertices are convex, therefore all stresses in the lattice must be of the same sign, and the lattice can be an equilibrated spider web. A chain of results that span a century allow us to see equilibrated spider webs as SVP's, and conversely. The alert reader may have noticed that equilibrated foams can be seen as a kind of "generalized" spider webs, in which the pressure difference between cells is the new ingredient, and is equilibrated by the curvature of the interfaces. It therefore turns out to be no surprise that these generalized spider webs (foams) are equivalent to an appropriate generalization of SVP's, namely, SMVP's, which include a multiplicative constant that gives rise to curved interfaces.

ACKNOWLEDGMENTS

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APPENDIX A: RECOGNITION PROBLEM FOR RECTILINEAR PARTITIONS

A classification of space into cells Ω_i separated by rectilinear interfaces Γ_{ij} is called a *rectilinear partition* of space.

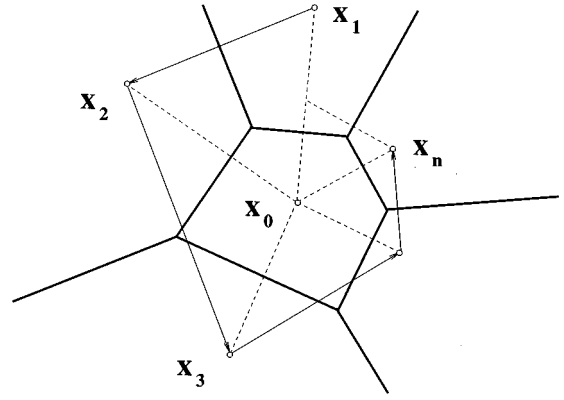


FIG. 16. An arbitrary rectilinear partition will not in general admit a reciprocal figure. The one in this example does not admit a reciprocal figure, and therefore cannot be a sectional Voronoï partition.

Given a rectilinear partition \mathcal{P} of the plane, a *reciprocal figure* $\mathcal{R}(\mathcal{P})$ is a planar graph composed of sites i joined by edges (ij) satisfying [20,15] the conditions that (a) sites i of \mathcal{R} are in one-to-one correspondence with cells Ω_i of \mathcal{P} , (b) edges (ij) of \mathcal{R} are in one-to-one correspondence with interfaces Γ_{ij} of \mathcal{P} , and (c) edges (ij) of \mathcal{R} are *normal* to interfaces Γ_{ij} of \mathcal{P} .

Clearly a reciprocal figure is defined up to arbitrary global dilatations and translations, since angles are not changed by them. Therefore, if \mathcal{P} admits a reciprocal figure, there are three degrees of freedom in its determination [20,15]. An arbitrary partition \mathcal{P} will *not* in general admit a reciprocal figure. We can see this with a simple example. Draw an arbitrary polygonal cell Ω_0 with n faces, and take arbitrary, rectilinear interfaces between its n neighbors (Fig. 16). Now take an arbitrary point x_0 on the plane to start with, and assign it to the central cell. This starting point is arbitrary, since a reciprocal figure is defined up to arbitrary translations. The other n points $\{x_1, \dots, x_n\}$ associated to the external cells must be somewhere on the n rays r_i stemming from x_0 and normal to the faces Γ_{0i} of Ω_0 . The global length scale is also arbitrary, so that, say, $\overline{x_0 x_1}$ can be freely chosen. Then we choose x_1 somewhere on r_1 . Now point x_2 is determined as the intersection of ray r_2 with the normal to face Γ_{12} going through x_1 . This can be repeated to obtain all n external points, but in general the figure will not *close*, that is, segment $\overline{x_n x_1}$ will not be normal to interface Γ_{1n} , whose orientation is arbitrary.

The planar graph formed by joining the sources P_i of a SVP with edges (ij) , one for each nonempty interface Γ_{ij} , constitutes a reciprocal figure for the SVP. Therefore every SVP has a reciprocal figure.

Recently Ash and Bolker showed that the existence of a reciprocal figure satisfying *orientation* [15] is also a *sufficient* condition for \mathcal{P} to be a SVP. A reciprocal figure \mathcal{R} satisfies orientation if for each bond ij , the sites i and j are oriented in the same way as cells Ω_i and Ω_j are with respect to Γ_{ij} . \mathcal{P} admits an oriented reciprocal figure $(\Leftrightarrow) \mathcal{P}$ is a SVP. This result solves the recognition problem for SVP's. The orientation condition can only be satisfied if all vertices of \mathcal{P} are *convex*. We will say that a vertex is

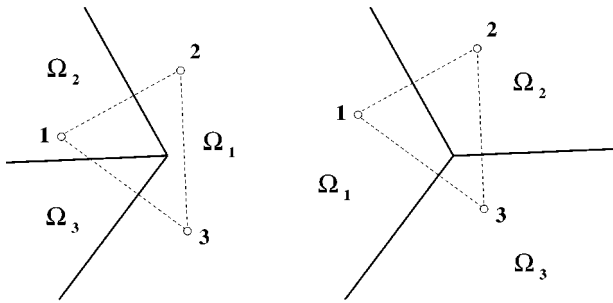


FIG. 17. Both these partitions with three cells admit a reciprocal figure. The one on the left does not satisfy orientation and therefore cannot be part of a sectional Voronoï partition. The one on the right satisfies orientation, and therefore it is a SVP. It is then possible to find sources f_1, f_2 , and f_3 in three-dimensional space, such that this partition is obtained as the cut $z=0$ of a three-dimensional Voronoï partition with respect to $\{f_i\}$. Sources f_i will be located at heights z_i above points $\{1,2,3\}$ in this figure.

convex if the internal angles formed by the interfaces are all smaller than π . Figure 17 shows two partitions with three cells. One of them is convex, the other is not. In the second case it is not possible to find a reciprocal figure that satisfies orientation, and therefore it cannot be a SVP.

Given a rectilinear partition \mathcal{P}^* , and an oriented reciprocal figure for it (as in Fig. 18), it is always possible [15] to find sources $\{f_i\}$ in three dimensions, located at heights z_i above the points P_i , such that \mathcal{P}^* is the section with Π_z of a three-dimensional VP with respect to $\{f_i\}$. The procedure to determine the heights z_i can be easily described. Vertices v_{ijk} of \mathcal{P}^* will be equidistant from sources f_i, f_j , and f_k (see Fig. 19). Start from an arbitrary vertex, say, v_{ijk} , and draw a spherical surface of arbitrary radius $r_{ijk}=r_0$ with center at that vertex. Now define sources f_i, f_j , and f_k as the intersections of this surface with the verticals (normals to Π_z) through P_i, P_j , and P_k , respectively. Next go to a neighboring vertex, which shares two sources with this one. Let us call it v_{ijm} . For this vertex, only source f_m has to be located, since f_i and f_j are known. Draw a spherical surface

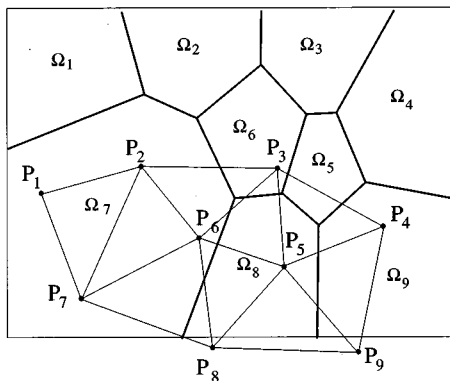


FIG. 18. A convex rectilinear partition (thick lines) with cells $\{A, B, \dots, I\}$ and its associated reciprocal figure (thin lines) with sites $\{a, b, \dots, i\}$ (black dots). This reciprocal figure (or any other figure obtained from dilatation and translation of this one) satisfies orientation, and therefore the partition is a sectional Voronoï partition. Sources of this SVP are located above points P_i .

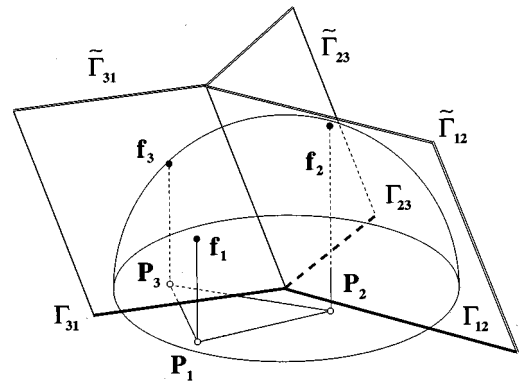


FIG. 19. Given an oriented reciprocal figure $\{P_1, P_2, P_3\}$ (white dots) for a rectilinear partition (interfaces Γ_{ij}), the sources f_i in three-dimensional space can be located above the points P_i . These sources must be equidistant from the vertex, therefore they are at the intersections (black dots) of a spherical surface centered at the vertex v_{123} with the normals through P_i .

with center v_{ijm} and containing f_i and f_j . Both will be simultaneously contained, since interface Γ_{ij} , on which v_{ijm} is located, is equidistant from f_i and f_j . The radius r_{ijm} of this surface is determined by the locations of f_i and f_j , which in turn are fixed by r_0 . Its intersection with the vertical n_m through P_m determines f_m . If this spherical surface does not intersect n_m , just choose a larger value of r_0 and start all over again (from the initial vertex). The construction proceeds in this manner until all sources have been determined. As mentioned above, the initial value of r_0 is tentative, in the sense that it may have to be modified (increased) if, at some point during the procedure, a normal line is not cut by the corresponding spherical surface from the vertex. It is easy to see that increasing the value of the starting radius r_0 is always enough to solve this problem. There is thus one degree of freedom in this construction (r_0). We conclude that, given a two-dimensional partition \mathcal{P} that admits a reciprocal figure, there is a four-parametric family of source locations such that \mathcal{P} is a SVP with respect to them. Three degrees of freedom come from the determination of the reciprocal figure itself (since a dilatation and/or translation of a reciprocal figure is again a reciprocal figure) and the last one from r_0 . This last degree of freedom results from the fact that a SVP is invariant if all heights are changed according to $z_i^2 \rightarrow z_i^2 + \alpha^2$ with α arbitrary [see Eqs. (3) and (4)].

Reciprocal figures were first studied by Maxwell [20,28] in relation to the rigidity of bar-joint frameworks in the plane. Maxwell pointed out that frameworks that have a reciprocal figure are able to support a self-stress, and conversely. The reason is that the edges of the reciprocal figure can be taken to represent forces transmitted by the edges of the original framework (rotated by $\pi/2$). Since these edges form closed polygons, the existence of a reciprocal figure implies the existence of an equilibrated set of internal stresses in the absence of an external load. The addition of the orientation condition (a condition not required by Maxwell's definition of a reciprocal figure) has the statical consequence that all signs of the stresses are equal, for example all traction or all compression. It is clear that no equilibrium is possible in the case of Fig. 17(a) if all three stresses are to

have the same sign. Figure 17(b), on the other hand, can be in equilibrium under compression or traction on all three interfaces. The conclusion is that every SVP is an equilibrium configuration for a spider web [15], and conversely, each such equilibrium configuration is a SVP. The existence of a reciprocal figure has also projective consequences, which have been studied by Crapo [29] and Whiteley [30].

APPENDIX B: INVERSION TRANSFORMATION

Here we briefly describe a geometric transformation called *inversion* [18]. We will find it extremely useful for our purpose of discussing circular partitions. An inversion with radius k around a point O located at \vec{r}_0 transforms a point P located at $\vec{r}_0 + \vec{r}$ into a point P' at $\vec{r}_0 + \vec{r}'$ satisfying

$$\vec{r}' = \frac{k^2}{r^2} \vec{r}, \quad (\text{B1})$$

where $r = |\vec{r}|$. The sphere of radius k and centered at O is invariant under this transformation, while the inside and outside of this sphere are interchanged. Obviously this transformation is self-inverse. Let us now describe some of the properties of this transformation in two dimensions [18]. Most of them apply trivially in higher dimensions.

(i) Circles not through O are transformed in circles not through O .

(ii) Circles through O are transformed into straight lines not through O .

(iii) Straight lines not through O are transformed in circles through O .

(iv) Straight lines through O are invariant.

(v) Angles are preserved (in modulus) by the inversion.

Given two points P_1 and P_2 at distances r_1 and r_2 from the inversion center, the distance d_{12} between them transforms as

$$d'_{12} = d_{12} \frac{k^2}{r_1 r_2}. \quad (\text{B2})$$

Using this result, we can easily see that the inversion is a ‘‘symmetry’’ of a MVP in any dimension if the intensities are also appropriately transformed [15]. More precisely, a MVP of R^n with respect to sources $\{P_{ij}\}$ with intensities $\{a_{ij}\}$ is transformed by an inversion into a MVP with respect to $\{P'_{ij}\}$ with intensities $\{a'_{ij}\}$, where the intensities satisfy

$$a'_i = \frac{a_i}{r_i} A_0. \quad (\text{B3})$$

Here A_0 is an arbitrary prefactor, the same for all a'_i 's, and r_i is the distance between source i and the inversion center O . To see this, it suffices to demonstrate that if $x \in \Gamma_{ij}$ then after an inversion, $x' \in \Gamma'_{ij}$, which is easily done using Eqs. (6), (B2), and (B3). The inversion transformation is of course also a symmetry of the SMVP (Sec. II D), if the inversion center O is on the cutting plane Π_z , since in this case the inversion leaves this plane invariant.

If the inversion center O happens to be located on an interface Γ_{ij} of a MVP (initially a spherical surface), the transformed interface Γ'_{ij} is a plane not through O . The resulting interface thus corresponds to a Voronoi partition with respect to sources i and j in their new locations. Therefore the transformed intensities a'_i and a'_j have to be equal after the inversion, which is verified using Eq. (B3),

$$O \in \Gamma_{ij} \Rightarrow \frac{a_i}{r_i} = \frac{a_j}{r_j} \Rightarrow a'_i = a'_j. \quad (\text{B4})$$

In the same way a SMVP with respect to two sources i and j is transformed into a SVP if $O \in \Gamma_{ij}$. The intensities a_i are transformed according to Eq. (B3), where r_i is the distance between O and source f_i in three-dimensional space. The way in which heights z_i transform is easily found using Eq. (B1). Notice that if the inversion center coincides with the location of a conjugate vertex v^* , then the transformed partition has a rectilinear vertex, since three interfaces are simultaneously transformed into straight lines. We will use this property of the inversion transformation when solving the recognition problem for SMVP's in Sec. III B.

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